Question 1. (15 marks)

Marks

a) Find $\int \frac{1}{6x-x^2}\,dx$. **2**

b) Find the exact value of k if $\int_1^{k^2} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 6e$. **4**

c) If $I_n = \int_1^e x(\log x)^n\,dx$ where n is a positive integer

i) Show that $I_n = \frac{e^2}{2} - \frac{n}{2} I_{n-1}$. **3**

ii) Hence evaluate $\int_1^e x(\log x)^3\,dx$. **2**

d) Find $\int_0^{\frac{\pi}{3}} \frac{dx}{2+\cos 2x}$ using the substitution $u = \tan x$. **4**

Question 2 *(15 marks)* Start the question in a new booklet.

a) Given $z = 1 - i\sqrt{3}$

i) Write z in modulus - argument form. **1**

ii) Hence find z^8 in the form $x+iy$ where x and y are real. **2**

iii) Find the least positive value of n such that z^n is real. **2**

b) Sketch each of the following regions on a separate Argand diagram

i) $|z-2-i| \le 2$ **2**

ii) $0 < \arg\left[(1+i)z\right] \le \frac{\pi}{2}$ **2**

c) OABC is a square. O represents the complex number 0

A represents the complex number $3+i$, B represents a complex number z and

C represents the complex number w.

D is the point of intersection of the diagonals.

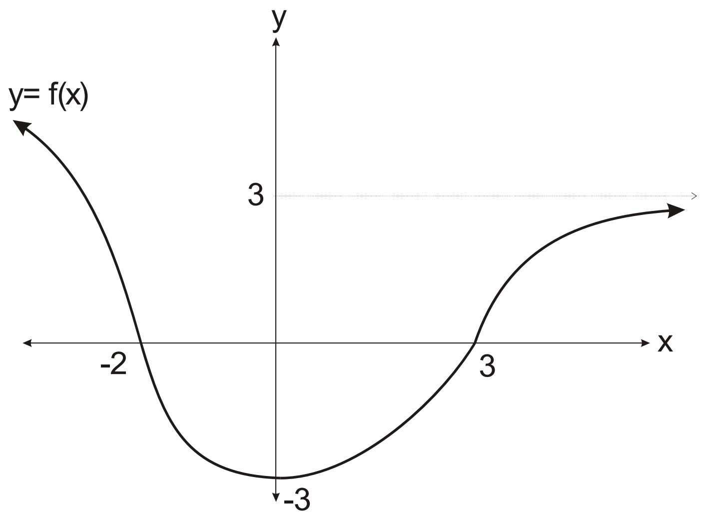
i) Find the complex numbers corresponding to points C and D in the form $x+iy$. **2**

ii) Find $\arg\left(\frac{w}{z}\right)$. **2**

iii) If E is the fourth vertex of the parallelogram OAEB find the complex number corresponding to E. **2**

Question 3 *(15 marks)* Start the question in a new booklet. **Marks**

a) The diagram shows the graph of $y = f(x)$



Sketch graphs of

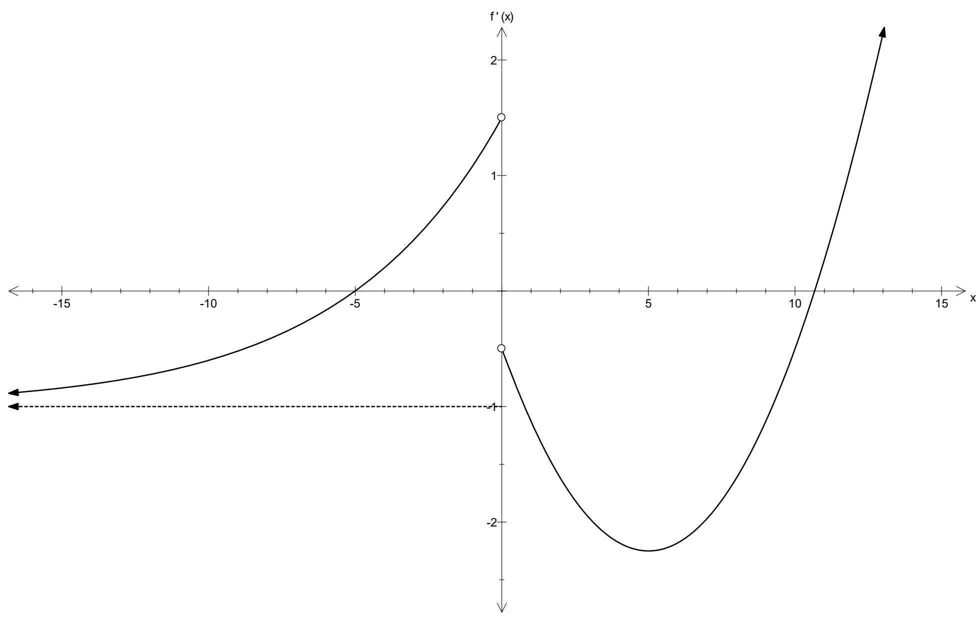
i) $y = |f(x)|$ **1**

ii) $y = \dfrac{1}{f(x)}$ **2**

iii) $y^2 = f(x)$ **2**

iv) the inverse function $y = f^{-1}(x)$ **2**

b) The diagram is a sketch of $y = f'(x)$ with a horizontal asymptote at $y = -1$. **3**
Sketch $y = f(x)$ given that it is continuous and $f(-15) = f(5) = 0$.
Clearly label important features.



Question 3 continued **Marks**

c) The hyperbola $xy = c^2$ touches the circle $(x-1)^2 + y^2 = 1$ at the point Q.

i) Show this information on a sketch. **1**

ii) Deduce that the equation $x^2(x-1)^2 + c^4 = x^2$ has a repeated real root $\beta > 0$ and two complex roots. **2**

iii) Find the value of β, c^2. **2**

Question 4 *(15 marks)* Start the question in a new booklet.

a) Given P is a point on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with foci S,S'.

i) Prove $|PS - PS'| = 2a$. **3**

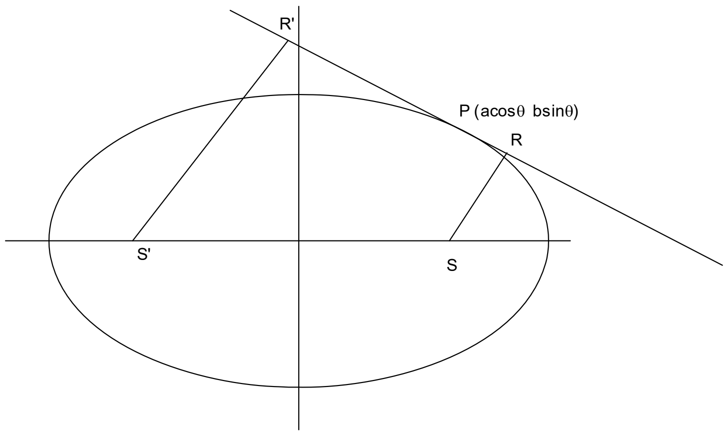
ii) If this hyperbola is rectangular prove $PS.PS' = OP^2$. **3**

b) Akram estimates that the probability of his winning any one game of tennis against a particular opponent is $\frac{1}{3}$. How many games should they play so that the probability that Akram wins at least one game is greater than 0.9? **4**

c) Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$,

i) Show that the tangent to the ellipse at the point P $(a\cos\theta, b\sin\theta)$ has equation $bx\cos\theta + ay\sin\theta - ab = 0$. **2**

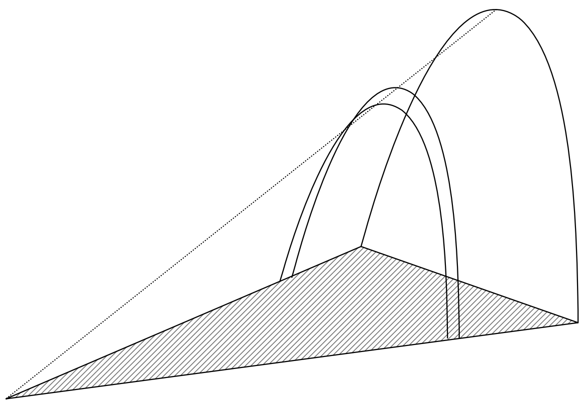
ii) R and R' are the feet of the perpendiculars from the foci S and S' respectively onto the tangent at P. Show that $SR.S'R' = b^2$. **3**



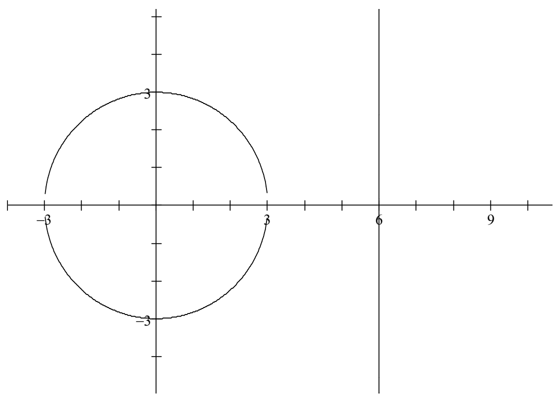
Question 5. *(15 marks)* Start the question in a new booklet. **Marks**

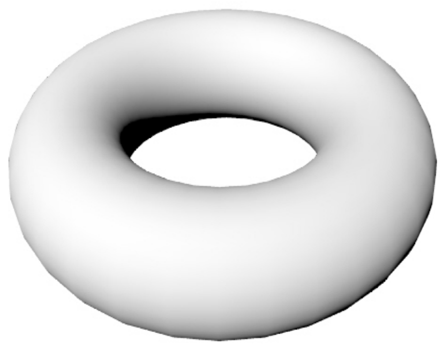
a) Show that the area bounded by the parabola $x^2 = 4ay$ and the latus rectum $y = a$ is equal to $\dfrac{8a^2}{3}$. **3**

b) A particular solid has a triangular base with all sides 6 metres. **6**
Cross sections taken parallel to one side of the base are parabolas. Each parabolic cross-section is such that its latus rectum lies in the base of the solid.
Find the volume of the solid.



c) The circle with equation $x^2 + y^2 = 9$ is rotated about the line $x = 6$ to form a torus.





i) Show, using the method of cylindrical shells that the volume V of the torus is given by **3**
$V = 4\pi\int_{-3}^{3}(6-x)\sqrt{9-x^2}\,dx$.

ii) Hence find the volume of the torus. **3**

Question 6. *(15 marks)* Start the question in a new booklet. **Marks**

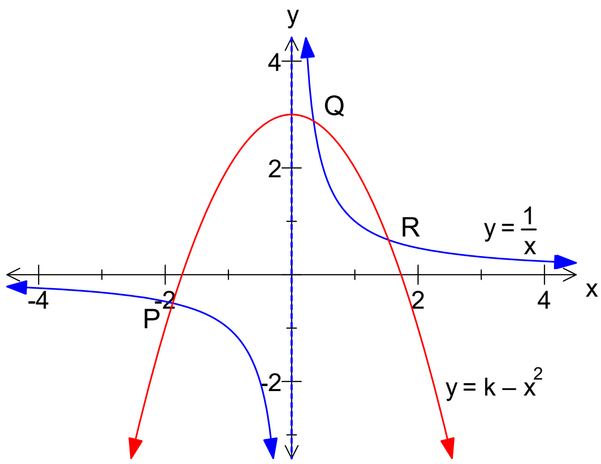
a) i) By considering factors, or otherwise show, that the roots of $z^6 - z^3 + 1 = 0$ are among the roots of $z^9 + 1 = 0$. **1**

ii) By selecting the appropriate roots of $z^9 + 1 = 0$, or otherwise, show that **4**

$$z^6 - z^3 + 1 = \left(z^2 - 2z\cos\frac{\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{5\pi}{9} + 1\right)\left(z^2 - 2z\cos\frac{7\pi}{9} + 1\right).$$

iii) Show that $\cos\frac{\pi}{9}\cos\frac{5\pi}{9} + \cos\frac{5\pi}{9}\cos\frac{7\pi}{9} + \cos\frac{7\pi}{9}\cos\frac{\pi}{9} = -\frac{3}{4}$. **3**

b)



The curves $y = k - x^2$ for some real k, and $y = \frac{1}{x}$ intersect at the points P, Q and R where $x = \alpha, x = \beta$ and $x = \gamma$ respectively.

i) Show that the monic equations with coefficients in terms of k whose roots are $\alpha^2, \beta^2, \gamma^2$ is given by $x^3 - 2kx^2 + k^2x - 1 = 0$. **3**

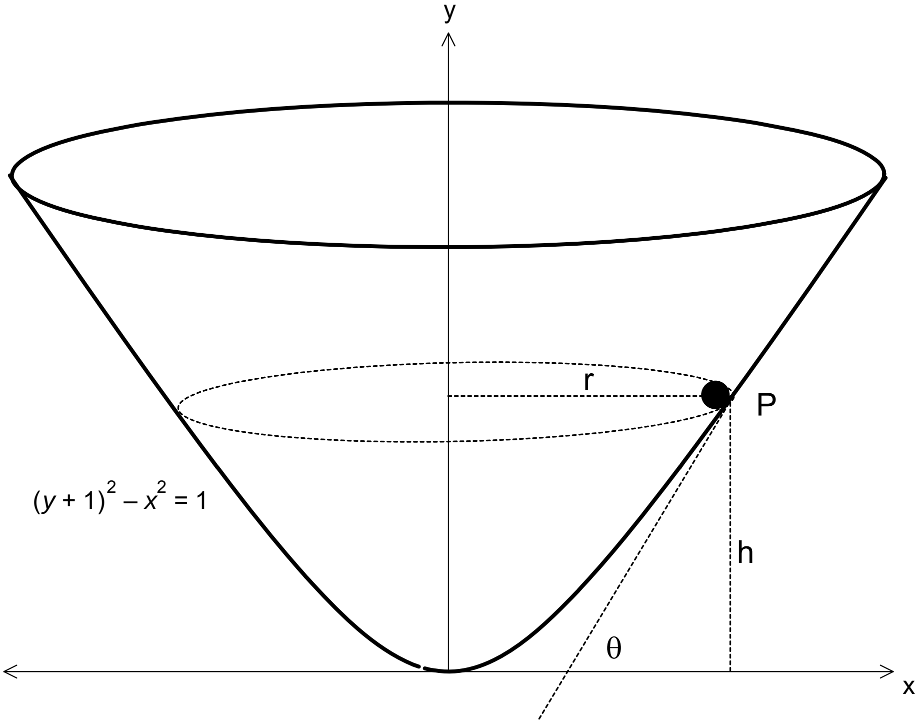
ii) Show the monic equation, with coefficients in terms of, whose roots are $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$ and $\frac{1}{\gamma^2}$ is $x^3 - k^2x^2 + 2kx - 1 = 0$. **2**

iii) Hence show that $OP^2 + OQ^2 + OR^2 = k^2 + 2k$, where O is the origin. **2**

Question 7. *(15 marks)* Start the question in a new booklet.

Marks

a)



A smooth bowl is formed by rotating the hyperbola $(y+1)^2 - x^2 = 1$ around the y axis.
A particle P of mass $m\ kg$ travels around the inside with constant angular velocity ω radians per second in a horizontal circle of radius r metres at a height h metres above the bottom of the bowl. The acceleration due to gravity is $g\ ms^{-2}$.

i) Show that if the tangent to the hyperbola $(y+1)^2 - x^2 = 1$ at the point (x_1, y_1) makes an angle θ with the positive x axis, then $\tan\theta = \dfrac{x_1}{1+y_1}$. **1**

ii) Draw a diagram showing the forces for P. **1**

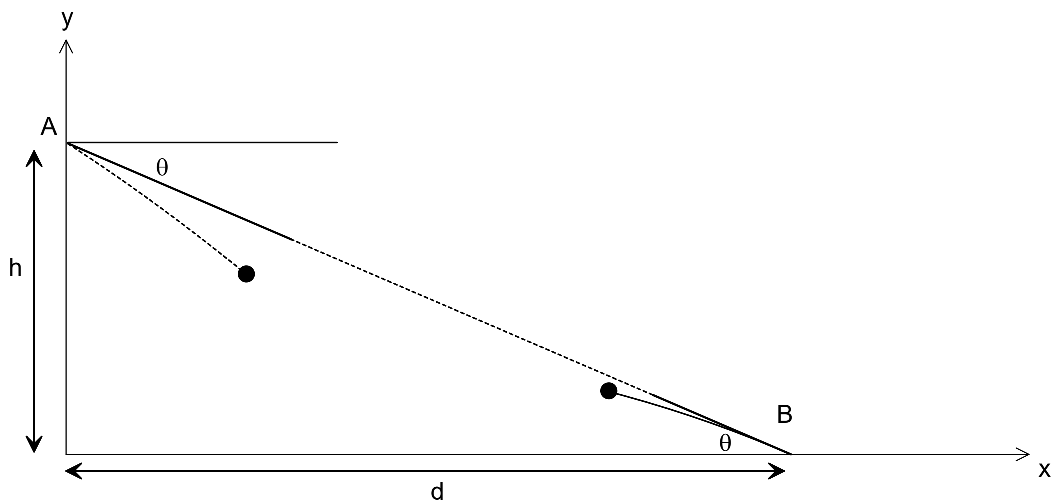
iii) Show $\omega^2 = \dfrac{g}{1+h}$. **2**

iv) Show that the force N Newtons exerted by the particle P on the bowl is given by $N = mg\sqrt{2 - \dfrac{1}{(1+h)^2}}$. **2**

Marks

Question 7 continued

b) The diagram shows the point A at a height h vertically above the point O. It also shows the point B which is positioned at a horizontal distance d from O.
A projectile is fired from A directly at point B with a Velocity V. At the same instant a projectile is fired from B directly at the point A with the same velocity V.



Let θ be the angle between the horizontal and the angle of projection.

Show that

i) The equations of motion of the two projectiles are **4**

$$x_A = Vt\cos\theta \qquad x_B = d - Vt\cos\theta$$

$$y_A = h - Vt\sin\theta - \frac{gt^2}{2} \qquad y_B = Vt\sin\theta - \frac{gt^2}{2}$$

ii) Show the particles will always meet . **3**

iii) Show that the height H at which they meet is given by **2**

$$H = \frac{h}{2} - \frac{g\left(h^2 + d^2\right)}{8V^2}.$$

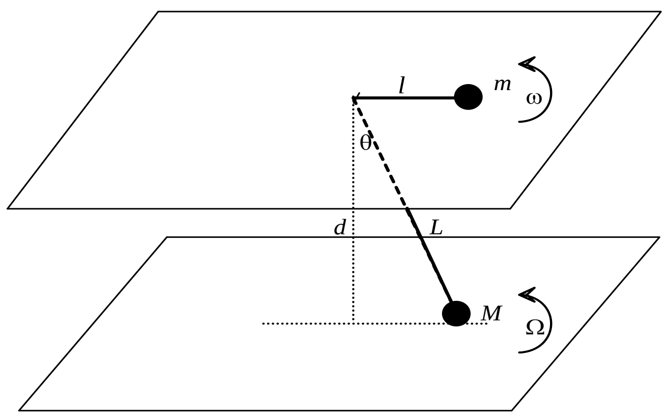
Question 8. *(15 marks)* Start the question in a new booklet. **Marks**

a) Given that $\sin^{-1} x$, $\cos^{-1} x$ and $\sin^{-1}(1-x)$ are acute.

i) Show that $\sin\left(\sin^{-1} x - \cos^{-1} x\right) = 2x^2 - 1$ **2**

ii) Solve the equation $\sin^{-1} x - \cos^{-1} x = \sin^{-1}(1-x)$. **3**

b)



Two particles are connected by a light inextensible string which passes through a small hole with smooth edges in a smooth horizontal table. One particle of mass m travels in a circle on the table with constant angular motion ω. The second particle of mass M travels in a circle with constant angular velocity Ω on a smooth horizontal floor distance d below the table. The lengths of string on the table and below the table are l and L respectively and L makes an angle θ with the vertical. **10**

i) draw diagrams showing the forces acting on each particle.

ii) If the floor exerts a force N on the lower particle, show $N = M\left(g - d\Omega^2\right)$.
State the maximum possible value of Ω for the motion to continue as described. What happens if Ω exceeds this value?

iii) By considering the tension force in the string, show $\frac{L}{l} = \frac{m}{M}\left(\frac{\omega}{\Omega}\right)^2$

iv) If the lower particle exerts zero force on the floor, show that the tension T in the string is given by $T = \frac{mgL}{d}$.

v) Given the table is 80cm high and the string is 1.5m long, while the masses on the table and on the floor are 0.4kg and 0.2kg respectively. The particles are observed to have the same angular velocity. If the lower particle exerts zero force on the floor, find, in terms of g the tension in the string.

END OF PAPER

Trial Ext2 2007

SOLUTIONS

Question 1.

a) $\int \frac{1}{6x-x^2}dx = \int \frac{dx}{9-(x-3)^2}$

$= \frac{1}{6}\log\frac{x}{6-x}$

b) $\int_1^{k^2} \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = \left[2e^{\sqrt{x}}\right]_1^{k^2}$

$= 2e^{|k|} - 2e = 6e$

$e^{|k|} = 4e$

$k = \pm\ln 4e$

c) i) $I_n = \int_1^e x(\ln x)^n dx = \int_1^e (\ln x)^n \frac{d}{dx}\left(\frac{x^2}{2}\right)dx$

$= \left[\frac{x^2}{2}\ln x\right]_1^e - \int_1^e \frac{x^2}{2}n(\ln x)^{n-1}\times\frac{1}{x}dx$

$= \frac{e^2}{2} - \frac{n}{2}\int_1^e x(\ln x)^{n-1}dx$

$= \frac{e^2}{2} - \frac{n}{2}I_{n-1}$

ii) $I_0 = \int_1^e x\,dx = \frac{e^2}{2} - \frac{1}{2}$

$I_1 = \frac{e^2}{2} - \frac{1}{2}I_0 = \frac{e^2}{4} + \frac{1}{4}$

$I_2 = \frac{e^2}{2} - I_1 = \frac{e^2}{4} - \frac{1}{4}$

$I_3 = \frac{e^2}{2} - \frac{3}{2}\left(\frac{e^{2-}}{4} - \frac{1}{4}\right)$

$= \frac{e^2}{8} + \frac{3}{8}$

d) $\int_0^{\frac{\pi}{3}} \frac{1}{2+\cos 2x}dx$

let $u = \tan x$ $\qquad \cos 2x = \frac{1-u^2}{1+u^2}$

$du = \sec^2 x\,dx$

$dx = \frac{du}{1+u^2}$

$x = 0 \Rightarrow u = 0$

$x = \frac{\pi}{3} \Rightarrow u = \sqrt{3}$

$= \int_0^{\sqrt{3}} \frac{du}{2+2u^2+1-u^2}$

$= \int_0^{\sqrt{3}} \frac{du}{3+u^2}$

$= \left[\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{u}{\sqrt{3}}\right)\right]_0^{\sqrt{3}}$

$= \frac{1}{\sqrt{3}}\times\frac{\pi}{4} - 0$

$= \frac{\pi\sqrt{3}}{12}$

Question 2.

a) i) $z = 1 - i\sqrt{3}$

$z = 2cis\left(-\frac{\pi}{3}\right)$

ii) $z^8 = 2^8 cis\left(\frac{-8\pi}{3}\right)$

$= 2^8 cis\left(\frac{-2\pi}{3}\right)$

$= 2^8\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -2^7 - 2^7\sqrt{3}i = -128 - 128\sqrt{3}\,i$

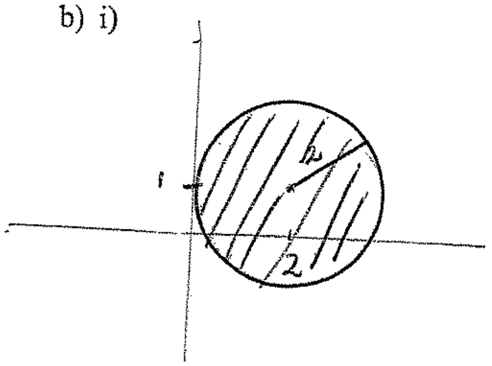
iii) $z^n = 2^n cis\left(-\frac{n\pi}{3}\right)$

real when $\frac{n\pi}{3} = k\pi$

for $k = 1 \quad n = 3$

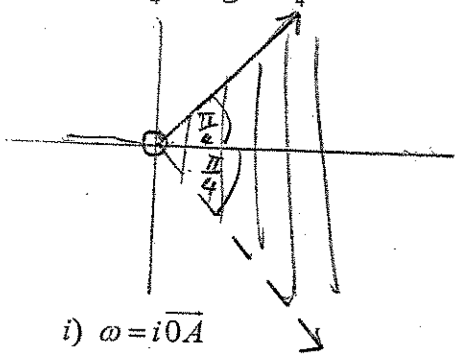
$\Rightarrow z^3 = -8$

b) i)

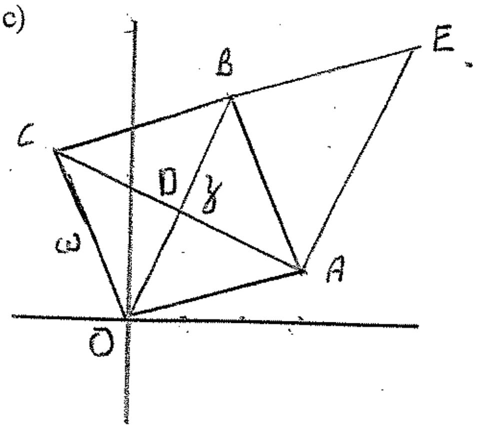


ii) $0 < arg\left(1+i\right) + arg\, z \leq \frac{\pi}{2}$

$-\frac{\pi}{4} < arg\, z \leq \frac{\pi}{4}$



c)



i) $\omega = i\overrightarrow{0A}$

$= -1+3i$

D is mid pt CA

$= (1, 2i)$

ii) $\arg\left(\frac{w}{z}\right) = \arg w - \arg z$

$= \text{B}\hat{\text{O}}\text{C} = \frac{\pi}{4}$

iii) $z = 2 \times \overline{\text{OD}}$

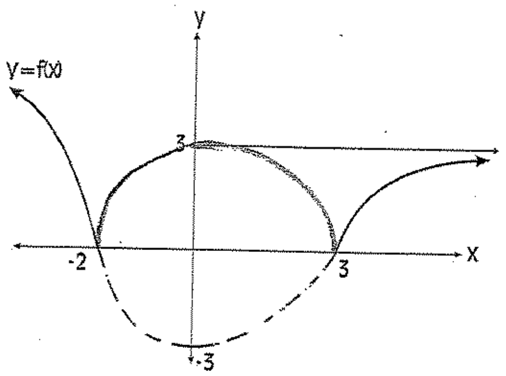
$\overrightarrow{\text{OE}} = \overrightarrow{\text{OA}} + \overrightarrow{\text{DB}}$

$= \begin{pmatrix}3\\1\end{pmatrix} + 2\begin{pmatrix}1\\2\end{pmatrix}$

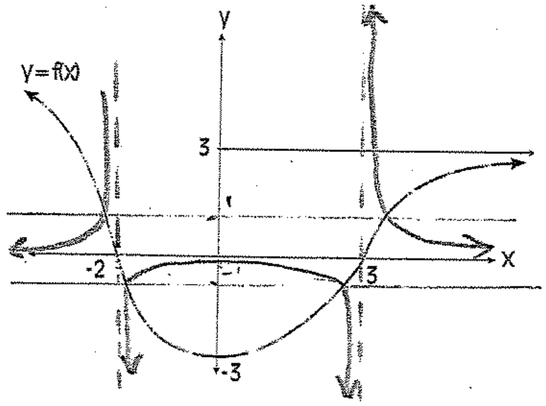
$= \begin{pmatrix}5\\5\end{pmatrix} \therefore$ E is $5+5i$

Question 3

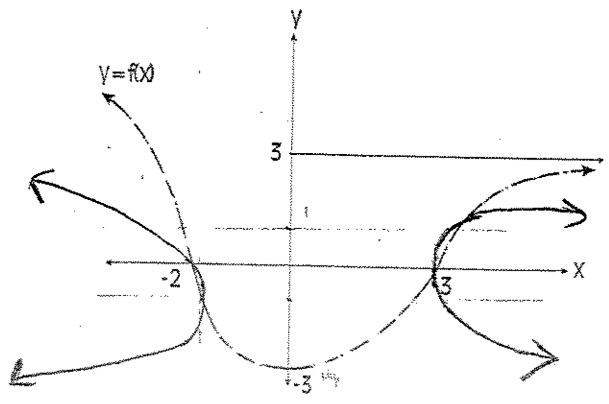
a)i)



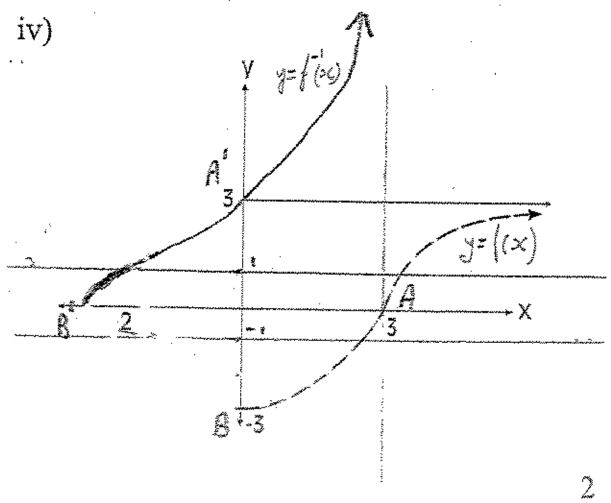
ii)



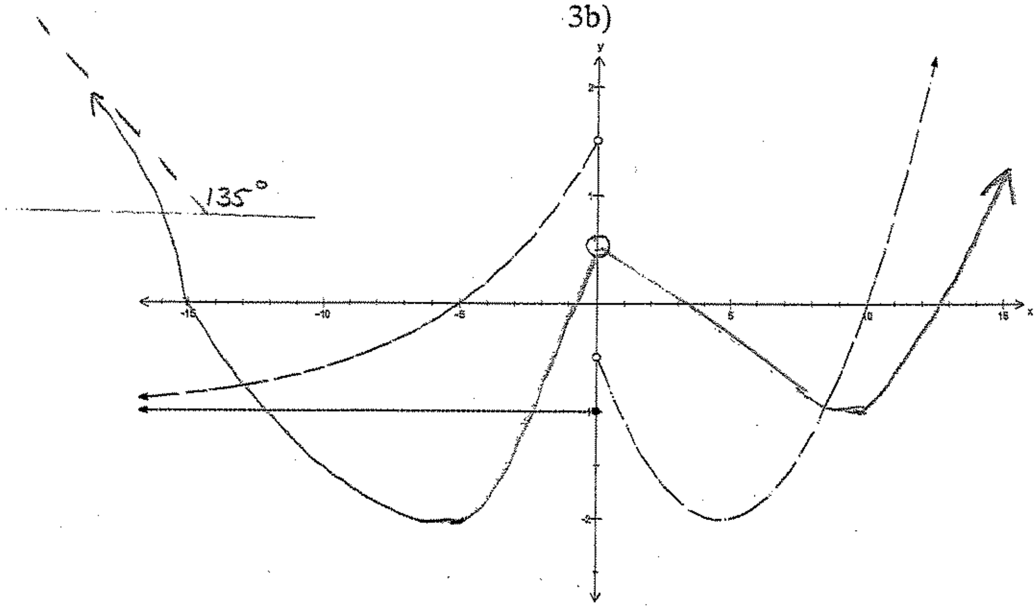
iii)



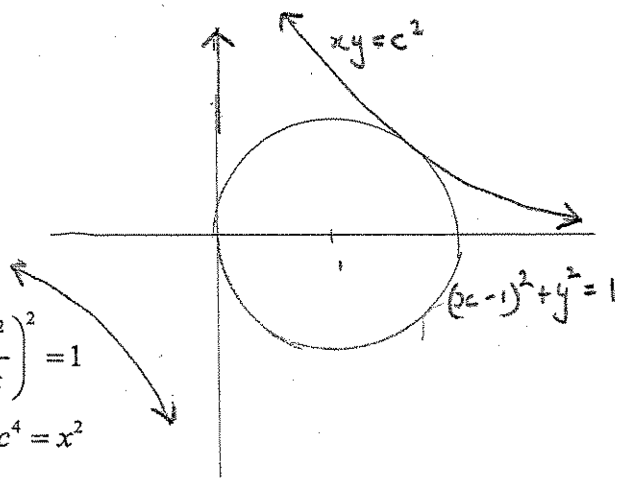
iv)



3b)



c) i) Diagram



ii) Touch when $(x-1)^2+\left(\frac{c^2}{x}\right)^2=1$

$$x^2(x-1)^2+c^4=x^2$$

There is clearly only one point of intersection. Since they touch in the 1st quadrant there must be a repeated root β with $\beta>0$.

β must have multiplicity either 2, or 4 since complex roots must occur in conjugate pairs.

Consider $P(x)=x^4-2x^3+c^4$

$$P'(x)=4x^3-6x^2$$

$$=0 \text{ when } x=\frac{3}{2},0 \text{ reject } 0$$

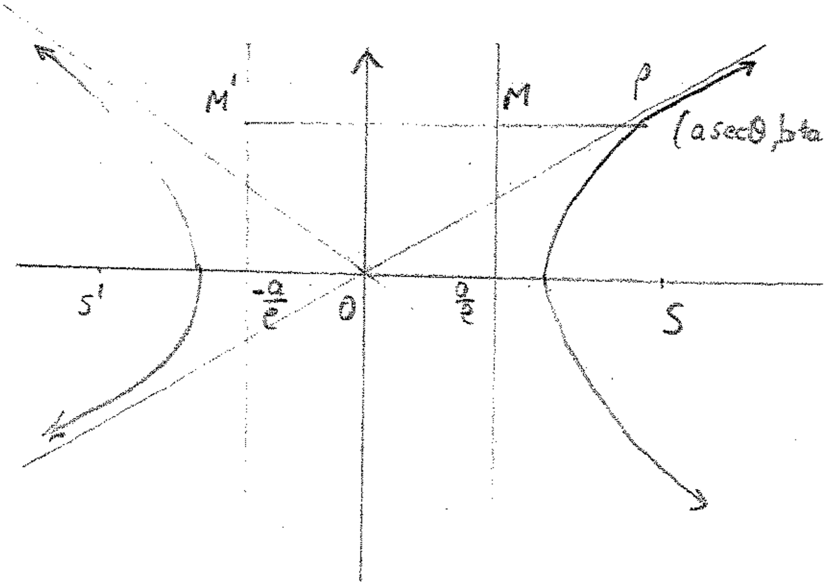
If the repeated root is of multiplicity 4 then $\sum\alpha=4\times\frac{3}{2}=6\neq-\frac{b}{a}\therefore$ not of multiplicity 4

$\therefore x=\beta=\frac{3}{2}$ is a double root only

$\therefore$ other roots must be complex.

OR $P''\left(\frac{3}{2}\right)\neq 0 \therefore$ only double root

Question 4.



a) i) $\dfrac{PS}{PM}=\dfrac{PS'}{PM'}=e$

$\therefore PS = ePM,\ PS' = ePM'$

$|PS\text{-}PS'| = e|PM' - PM|$

$= e \times 2 \times \dfrac{a}{e}$

$= 2a$

ii) Since rectangular $a = b,\ e = \sqrt{2}$

$OP^2 = a^2\sec^2\theta + a^2\tan^2\theta$

$= a^2\left(\sec^2\theta + \sec^2\theta - 1\right)$

$= a^2\left(2\sec^2\theta - 1\right)$

$PS.PS' = e^2\, PM.PM'$

$= e^2\left(a\sec\theta - a/e\right)\left(a\sec\theta + a/\sqrt{2}\right)$

$= 2a^2\sec^2\theta - a^2$

OR

$PS.PS^1 = a^2\left(e\sec\theta - 1\right)\left(e\sec\theta + 1\right)$

$= a^2\left(e^2\sec^2\theta - 1\right)$

$= a^2\left(2\sec^2\theta - 1\right)$

b) i) Let P (Akran wins) $= p, \quad q = 1 - p$

Let number of games $= n$ Consider $(p+q)^n = \Sigma\, {}^nC_r p^r q^{n-r}$

$P(\text{wins} \geq 1) = 1 - P(\text{wins no games})$

$= 1 - q^n$

let $1 - q^n > 0.9 \quad \therefore 0.1 > \left(\frac{2}{3}\right)^n$

$n > \frac{\ln 0.1}{\ln 2/3} \approx 5.68 \quad \therefore$ needs to play 6 games

4c) i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

diff^n gives $\frac{2x}{a^2} + \frac{2y\frac{dy}{dx}}{b^2} = 0$

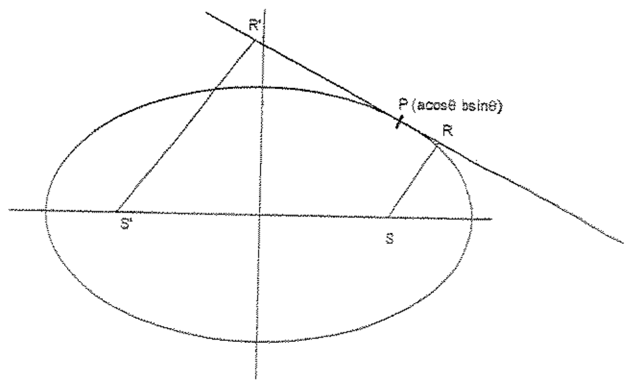
$\therefore \frac{dy}{dx} = \frac{-b^2x}{a^2y}$ at $(a\cos\theta, b\sin\theta) = -\frac{b}{a}\cot\theta$

$\therefore$ tangent is $y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$

$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$

$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta + \cos^2\theta) = ab$

ii) $SR.S'R'$



$= \dfrac{|b\cos\theta \times ae - ab| \times |-b\cos\theta \times ae - ab|}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}}$

$= \dfrac{|a^2b^2e^2\cos^2\theta - a^2b^2|}{b^2\cos^2\theta + a^2\sin^2\theta}$

$= \dfrac{b^2\left|\left(a^2 - b^2\right)\cos^2\theta - a^2\right|}{b^2\cos^2\theta + a^2\sin^2\theta}$

$= \dfrac{b^2\left|-b^2\cos^2\theta - a^2\sin^{-2}\theta\right|}{b^2\cos^2\theta + a^2\sin^2\theta} = b^2$

Question 5.

$a)\quad y=\frac{1}{4a}x^2$

Area under curve $= A = \frac{2}{4a}\int_0^{2a} x^2\,dx$

$= \frac{1}{2a}\left[\frac{x^3}{3}\right]_0^2 = \frac{4a^2}{3}$

Shaded area $= 4a \times a - \frac{4a^2}{3}$

$= \frac{8a^2}{3}$

$b)$ Consider slice at dist x from A

BC = AB tan 30 $= \frac{x}{\sqrt{3}}$

Now BC is semi latus rectum

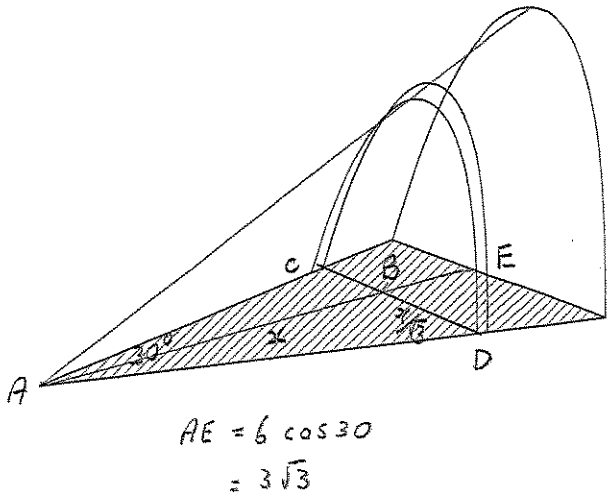
$\therefore a = \frac{x}{2\sqrt{3}}$

$A = \frac{8}{3}\times\frac{x^2}{4\times 3} = \frac{2x^2}{9}$

$\delta V = \frac{2}{9}x^2\delta x$

$$V = \lim_{\delta x\to 0}\sum_{x=0}^{3\sqrt{3}}\delta v = \frac{2}{9}\int_0^{3\sqrt{3}} x^2\,dx$$

$$= \frac{2}{9}\left[\frac{x^3}{3}\right]_0^{3\sqrt{3}} = \frac{2}{9}\times\frac{81\sqrt{3}}{3} = 6\sqrt{3}\,u^2$$



5c) Consider a vertical strip as shown.
Rotate this strip to form a cylindrical shell of height $2y$ and outer radius $(6-x)$

To 1st order approx this shell volume = volume of rectangular prism

$\therefore \delta V = 4\pi(6-x)y\delta x$

$$V = \lim_{\delta x\to 0}\sum_{x=-3}^{3}\delta v$$

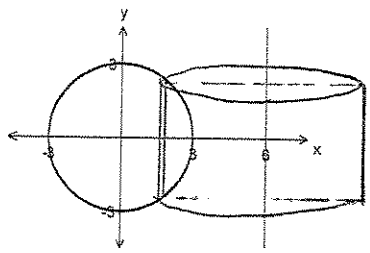
$$V = 4\pi\int_{-3}^{3}(6-x)y\,dx$$

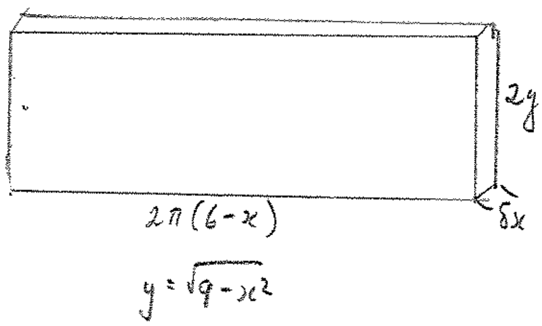
$$= 4\pi\int_{-3}^{3}(6-x)\sqrt{9-x^2}\,dx$$

$$= 4\pi\int_{-3}^{3}6\sqrt{9-x^2} - \int_{-3}^{3}x\sqrt{9-x^2}$$

$$= 24\pi\times\frac{\pi\times 3^2}{2} - 0 \quad \text{(since odd fn)}$$

$= 108\pi^2$





Question 6.

$a)$ $i)$ $z^9+1=(z^3+1)(z^6-z^3+1)$

$=0$ when $z^3+1=0$

or $z^6-z^3+1=0$

$\therefore$ roots of $z^6-z^3+1=0$ are the roots of $z^9+1=0$ that are not the roots of $z^3=-1$

$ii)$ The 9 roots of $z^9+1=0$ include -1 and are equally spaced around the unit circle in an Argard diagram by an angle of $\frac{2\pi}{9}$.

The roots -1 and those spaced at an angle of $\pm\frac{2\pi}{3}$ from -1 are the cube roots of -1.

The remaining roots are therefore the roots of $z^6-z^3+1=0$.

Arrange these roots in conjugate pairs $\frac{\pi}{9},\frac{-\pi}{9},\frac{5\pi}{9},\frac{-5\pi}{9},\frac{7\pi}{9},\frac{-7\pi}{9}$

Consider $(z-\alpha)(z-\overline{\alpha})=z^2-2z\Re(\alpha)+|\alpha|$

$$\therefore z^6-z^3+1=\left(z^2-2z\cos\frac{\pi}{9}+1\right)\left(z^2-2z\cos\frac{5\pi}{9}+1\right)\left(z^2-2z\cos\frac{7\pi}{9}+1\right)$$

$iii)$ Equate coeff of z^2 $LHS=0$

$$RHS=4\left(\cos\frac{\pi}{9}\cos\frac{5\pi}{9}+\cos\frac{\pi}{9}\cos\frac{7\pi}{9}+\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}\right)+3$$

$$\therefore\cos\frac{\pi}{9}\cos\frac{5\pi}{9}+\cos\frac{\pi}{9}\cos\frac{7\pi}{9}+\cos\frac{5\pi}{9}\cos\frac{7\pi}{9}=-\frac{3}{4}$$

$6b)$ $i)$ $y=k-x^2\quad-(1)$

$y=\frac{1}{x}\quad-(2)$

Solve (1) and (2) $\therefore 1=kx-x^3$

$x^3-kx+1=0$ has roots α,β,γ

let $x=\sqrt{y}$

$\therefore y^{\frac{3}{2}}-ky^{\frac{1}{2}}=$

square b.s. $\quad y^3-2ky^2+k^2y=1$

$\therefore x^3-2kx^2+k^2x-1=0$ has roots $\alpha^2,\beta^2,\gamma^2\quad-(A)$

$ii)$ Let $x=\frac{1}{y}$

$\therefore\frac{1}{y^3}-2k\frac{1}{y^2}+\frac{k^2}{y}-1=0$

$\therefore 1-2ky+k^2y^2-y^3=0$

$\therefore x^3-k^2x^2+2kx-1=0$

has roots $\frac{1}{\alpha^2},\frac{1}{\beta^2},\frac{1}{\gamma^2}\quad-(B)$

$iii)$ $OP^2=\alpha^2+\frac{1}{\alpha^2}$ similarly $OQ^2=\beta^2+\frac{1}{\beta^2}$, $OR^2=\gamma^2+\frac{1}{\gamma^2}$

$\therefore OP^2+OQ^2+OR^2=\alpha^2+\beta^2+\gamma^2+\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}$

$=2k+k^2$ by sum of roots from (A) and (B).

Question 7.

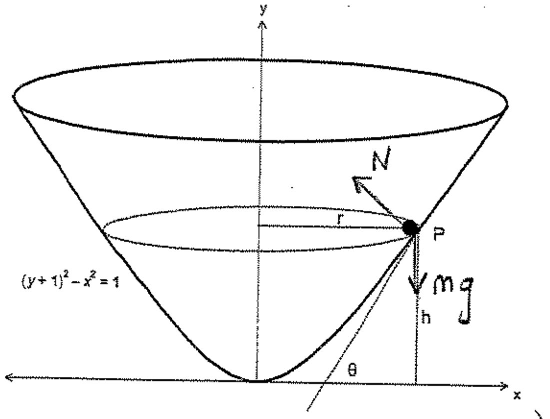
$a)i)$ $(y+1)^2 - x^2 = 1$

$$2(y+1)\frac{dy}{dx} - 2x = 0$$

$$\frac{dy}{dx} = \frac{x}{y+1}$$

$$\therefore \tan\theta = \frac{x_1}{1+y_1}$$

ii)



)

$iii)$ $H: N\sin\theta = mr\omega^2 \quad -(1)$

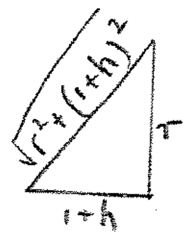
$V: N\cos\theta = mg \quad -(2)$

$$\therefore N = \frac{mg}{\cos\theta}$$

$$Sub-(1)\ \frac{mg}{\cos\theta}\sin\theta = mr\omega^2$$

$$\tan\theta = \frac{r}{1+h} \qquad \frac{g \times r}{(1+h)} = r\omega^2$$

$$\therefore \omega^2 = \frac{g}{1+h}$$



$iv)$ From (2) $N = \dfrac{mg}{\cos\theta}$ $\quad \tan\theta = \dfrac{r}{1+h}$

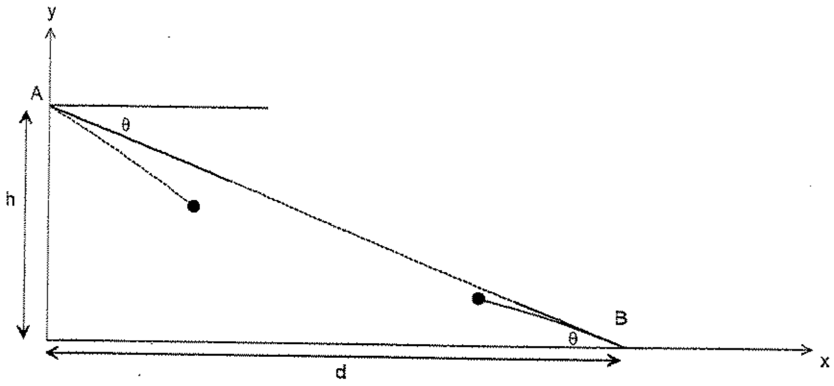
but (r,h) lies on hyperbola $\therefore r^2 = (1+h^2) - 1$

$$\cos\theta = \frac{1+h}{\sqrt{2(1+h)^2 - 1}}$$

$$\therefore N = \frac{mg}{(1+h)}\sqrt{2(1+h)^2 - 1}$$

$$= mg\sqrt{2 - \frac{1}{(1+h)^2}}$$

7b)



i) $\ddot{x}_A = 0$

$\therefore \dot{x}_A = C \quad t=0, \ \dot{x} = V\cos\theta$

$\therefore \dot{x}_A = V\cos\theta$

$x_A = V\cos\theta . t + c' \ t=0 \ x_A = 0 \therefore c' = 0$

$x_A = V\cos\theta . t$

$\ddot{y}_A = -g$

$\dot{y}_A = -gt + k \ \ t=0 \ \dot{y} = -V\sin\theta$

$\therefore \dot{y}_A = -gt - V\sin\theta$

$y_A = -\frac{1}{2}gt^2 - V\sin\theta . t + k' \ \ t=0 \ \ y_A = h \therefore k' = h$

$y_A = -\frac{1}{2}gt^2 - V\sin\theta . t + h$

$\ddot{x}_B = 0$

$\dot{x}_B = c'' \ \ t=0 \ \ \dot{x} = -V\cos\theta$

$\therefore \dot{x}_B = -V\cos\theta$

$x_B = -V\cos\theta . t + c''' \ \ t=0 \ \ x_B = d \therefore c''' = d$

$x_B = d - V\cos\theta . t$

$\ddot{y}_B = -g$

$\dot{y}_B = gt + k'' \ \ t=0 \ \ \dot{y}_B = V\sin\theta$

$\dot{y}_B = -gt + V\sin\theta$

$y_B = -\frac{1}{2}gt^2 + V\sin\theta . t + k''' \ \ t=0 \ \ y_B = 0 \therefore k''' = 0$

$y_B = -\frac{1}{2}gt^2 + V\sin\theta . t$

ii) when $x_A = x_B$

$t.V\cos\theta = d - t.V\cos\theta$

$t_x = \frac{d}{2V\cos\theta} \qquad -(1)$

when $y_A = y_B$

$-\frac{1}{2}gt^2 - tV\sin\theta + h = -\frac{1}{2}gt^2 + V\sin\theta t$

$t_y = \frac{h}{2V\sin\theta} \qquad -(2)$

now $\tan\theta = \frac{h}{d} \ \therefore d\sin\theta = h\cos\theta$

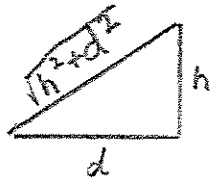
sub in (1) $\quad t_x = \dfrac{d}{2V\frac{d\sin\theta}{h}} = \dfrac{h}{2V\sin\theta} = t_y$

$\therefore$ The two particles have the same x, y coordinates at the same time.

Hence they meet.

iii) $H = -\frac{1}{2}gt^2 + V\sin\theta . t$

$tan\theta = \dfrac{h}{d}$



$$= -\frac{1}{2}g\frac{h^2}{4V^2\sin^2\theta} + V\sin\theta \times \frac{h}{2V\sin\theta}$$

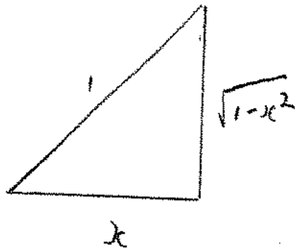
$$= -\frac{1}{2}g\frac{h^2}{4V^2 h^2 / (h^2+d^2)} + \frac{h}{2} = \frac{h}{2} - \frac{g(h^2+d^2)}{8V^2}$$

Question 8.

a) i) $L.H.S.\ \sin(\sin^{-1}x-\cos^{-1}x)$

$=x\times x-\sqrt{1-x^2}\times\overline{1-x^2}$

$=2x^2-1=R.H.S.$



ii) $\sin^{-1}x-\cos^{-1}x=\sin^{-1}(1-x)$

$\therefore 2x^2-1=1-x$

$2x^2+x-2=0$

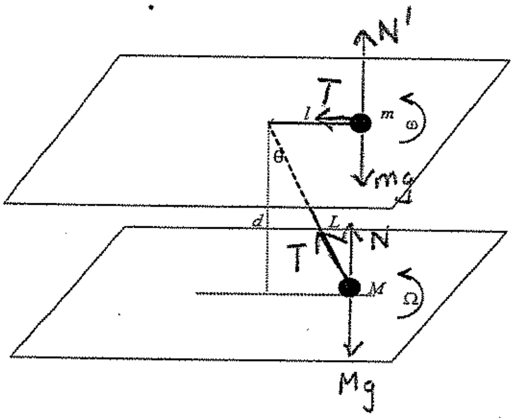
$x=\dfrac{-1\pm\sqrt{1+16}}{4}\approx 0.7808,-1.2808$

but $\sin^{-1}(1-x)$ must be acute

Now $1-(-1.2808)=2.2808^c$ which is not acute

$\therefore$ reject $x=-1.2808$ Solution $x=\frac{1+\sqrt{17}}{4}$

8b) i)



ii) Consider M: Hor: $T\sin\theta=MxL\sin\theta.\Omega^2$

Vert: $N+T\cos\theta=Mg$

$N=(Mg-T\cos\theta)$

$=Mg-ML\cos\theta.\Omega^2$

$=M\left(g-L\cos\theta.\Omega^2\right)$

$=M\left(g-d\Omega^2\right)$

$N\geq 0\quad\therefore\quad \Omega^2\leq\dfrac{g}{d}$

If $\Omega>\dfrac{g}{d}$ the mass M will lift off the surface.

iii) Consider m: Vert: $N'=mg$

Hor: $T=ml\omega^2$ $\quad -(2)$

Equate (1) & (2) $ML\Omega^2=ml\omega^2$

$\dfrac{L}{l}=\dfrac{m}{M}\left(\dfrac{\omega}{\Omega}\right)^2$ $\quad -(3)$

iv) If $N=0$ $\quad \Omega^2=\dfrac{g}{d}$ $\quad T=ML\Omega^2$ $\quad -(4)$

$=\dfrac{MLg}{d}$

v) $x=8\quad L+l=1.5\quad m=0.4\quad M=0.2$

from (3) $\dfrac{L}{l}=2\quad\therefore L=2l$

but $L+l=1.5$

$l=0.5,\ L=1$

$\therefore\ T=\dfrac{0.2\times 1\times g}{0.8}=\dfrac{g}{4}$